

Introducing spin to classical phase space

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Abstract

The kinematic degrees of freedom of spinning particles are analyzed and an explicit construction of the phase space and the symplectic structure that accomodates them is presented. A Poincare invariant theory of classical spinning particles that generalizes the work of Proca and Barut to arbitrary spin is given using spinor variables. Second quantization is naturally connected to the unphysical nature of zitterbewegung. Position variables can not be disentangled from spin in a canonical way, nor can the phase space be reduced to the usual description (x, p) and a vector spin.

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Classical Mechanics is based on the idea of the point particle endowed with mass. It successfully determines the evolution of the particle by the sole virtue of the Newton Principles. These have been formulated in a variety of ways of which in this paper we will use the economical and elegant Action Principle. Assuming that the action A describes the motion of a free particle through the Euler-Lagrange equations, we can impose $\delta A = 0$ for translations and Lorentz transformations to ensure that any pair of independent inertial observers will detect the same motion of the particle. That this gives the Galileo principle (or its equivalent, the First Newton Law) can be seen as follows: Be x^μ , \dot{x}^μ and τ the coordinates, velocities and proper time of the particle ($\dot{x}^\mu = dx^\mu/d\tau$). The invariance of the action implies

$$\delta A = (p_\mu \delta x^\mu)|_{\tau_1}^{\tau_2} = 0, \quad p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} \quad (1)$$

where $\delta x^\mu = \epsilon^\mu$ or $\delta x^\mu = \epsilon^\mu_\nu x^\nu$ for translations or Lorentz transformations respectively. From the above one obtains the conservation of momentum p_μ and angular momentum $L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$. In practice there are only seven independent conserved quantities p_μ and $L_{\mu\nu} p^\nu$, because $L_{\mu\nu}^* p^\nu$, where L^* is the dual of L , vanish by construction. The object $X_\mu = \frac{1}{p^2} L_{\mu\nu} p^\nu$ gives the part of x orthogonal to p , so the particle trajectory is simply:

$$x_\mu = X_\mu + f(\tau) p_\mu \quad (2)$$

where $f(\tau)$ is a scalar function of the proper time. From this one gets:

$$\frac{\vec{x}(\tau) - \vec{x}(0)}{x^0(\tau) - x^0(0)} = \frac{\vec{p}}{p^0} = \vec{v}, \quad (3)$$

which is the sought Galileo Principle written in the most accesible way to an inertial observer. Summarizing, the trajectory is a straight line completely determined by the six constants of motion $p_\mu/\sqrt{p^2}$ and X_μ . The particle travels along the trajectory with constant speed v . The physical requirement that τ be the proper time, i.e. $d\tau^2 = dx_\mu dx^\mu$, or $\dot{x}^2 = 1$, gives $\dot{f} = 1/m$, where $m = \sqrt{p^2}$ is the particle mass. In addition, the observer can set up time -adjusting it to the particle time- by fixing the initial condition $f(0)$. This derivation is independent of the explicit form chosen or the lagrangian, but a consequence of combining the action principle with the invariance under Poincare transformations.

Classical spinning particles are typically described [1] like points endowed with a mass and a spin angular momentum $S_{\mu\nu}$. Along with p the total angular momentum

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}. \quad (4)$$

is conserved. What can be known about particle kinematics for a generic $S_{\mu\nu}$? J is a constant, but we have not proved the same for L nor for S , we only

know that $\dot{S} = -\dot{L}$. The six constants of motion contained in J can be made explicit by

$$X_\mu = \frac{1}{p^2} J_{\mu\nu} p^\nu, \quad W_\mu = -J_{\mu\nu}^* p^\nu = -S_{\mu\nu}^* p^\nu \quad (5)$$

W is the Pauli-Lubansky vector describing the spin of the particle, while X is a fixed position in terms of which we can express the trajectory as

$$x_\mu(\tau) = X_\mu + f(\tau)p_\mu + r_\mu(\tau) \quad (6)$$

The vector r , the radius, is defined by

$$r_\mu = -\frac{1}{p^2} S_{\mu\nu} p^\nu. \quad (7)$$

Its origin can be traced back to the independent existence of orbital and internal angular momentum. This vector yields a tiny deviation of order (spin/mass) from the trajectory of the spinless free particle Eq. (2). It is not restricted to be a constant, opening the possibility for a non rectilinear $x(\tau)$ (a classical version of zitterbewegung). Sufficient coarsening will only reveal the “external” trajectory of (2) and possibly a finite constant spin W . Coarsening erases the internal angular momentum generated by zitterbewegung, the first term in

$$S_{\mu\nu} = -(r_\mu p_\nu - r_\nu p_\mu) + \frac{1}{p^2} \epsilon_{\mu\nu\rho\sigma} W^\rho p^\sigma. \quad (8)$$

In general one would need of additional subsidiary conditions to fix the time evolution of r . For instance, a constant spin modulus gives a constant norm radius:

$$\frac{1}{2} S_{\mu\nu} S^{\mu\nu} = -\frac{W^2}{p^2} + p^2 r^2 \quad (9)$$

An arbitrary antisymmetric tensor decomposes into the sum of two orthogonal planes. The condition that $S_{\mu\nu}$ corresponds to one and only one of these planes is

$$\frac{1}{2} S_{\mu\nu}^* S^{\mu\nu} = 2Wr = 0 \quad (10)$$

Finally, r is a spacelike fourvector of constant norm, whose only allowed motion would be a rotation in the plane π_r orthogonal to W and p . The particle would perform a helical motion composed of the uniform motion proper of the spinless particle and the rotation in the plane π_r . The helix thread number is not given yet. A particular case of (9) and (10) is $S_{\mu\nu} \dot{x}^\nu = 0$ which would add that information.

To describe the above situation beyond kinematics, one needs a lagrangian linear in velocities to avoid getting \dot{x} in terms of p . This is not enough still, since the spin degrees of freedom have to participate in the hamiltonian, and hence in the canonical formalism, in order to get a r dependent trajectory.

Here, we describe classical spinning particles in mathematical terms as belonging to irreducible representations of the Poincare group of mass m and spin s . We implement this idea in classical mechanics by including in the configuration space of the particle additional variables ξ and $\dot{\xi}$ (functions of the proper time) transforming according to the representation chosen. Invariance of the action under translations ($\delta\xi = 0$) will be achieved by momentum conservation, while Lorentz transformations ($\delta\xi = \epsilon(s)\xi$) will require the conservation of total angular momentum. The idea of using Dirac spinors in classical mechanics is originally due to Proca [2] and Schiller [3]. The latter author demonstrated that the hamiltonian proposed by Kramers [4] to describe the evolution of a classical dipole in an external electromagnetic field could be reformulated in terms of a Dirac bispinor. He then generalized the Hamilton-Jacobi formalism to include the motion of the dipole in a classical field theory framework. Proca intended to build a new point mechanics in terms of classical spinors to lay the foundations of a new quantization programme. Barut and collaborators gave a vigorous impulse to this proposal -promoted by them to a classical model of the Dirac electron [5]- and were able to show the emergence of QED from the path integral of this classical particle [6].

We now focus on the phase space of the spinning particle and recall that we are dealing with elementary systems, i.e. with irreducible representations of the Poincare Group [7]. The ξ 's will then be the spinors transforming according to the representation chosen. To the pair of variables ξ and its time derivative $\dot{\xi}$ we will associate a pair of canonical conjugate variables (ξ, η) , and enlarge the coordinates and momenta of the usual phase space to $P = \{(x^\mu, p_\mu), (\xi, \eta)\}$. Thus, each elementary system is characterized by the representation chosen for the phase space. P can be labeled with two indices (m, s) giving the mass and spin of the particle. We will form Lorentz scalars, vectors and tensors out of these by forming bilinears $\eta\Gamma_{\mu_1\cdots\mu_n}\xi$ and combinations of x and p in the standard form. We also form higher spinors from ξ or η . We can give a simple recipe for the symplectic structure on P : Given any pair of functions A, B of P , we define the canonical bracket as

$$\{A, B\} = \frac{\partial A}{\partial x^\mu} \frac{\partial B}{\partial p_\mu} - \frac{\partial A}{\partial p_\mu} \frac{\partial B}{\partial x^\mu} + \frac{\partial A}{\partial \xi} \frac{\partial B}{\partial \eta} - \frac{\partial B}{\partial \xi} \frac{\partial A}{\partial \eta} \quad (11)$$

where the order of the factors in the last term of the r.h.s. is chosen by notational convenience. It can be seen easily that the above fulfills all the conditions necessary to become an appropriate Poisson bracket [8].

We now require Poincare invariance:

1. Under translations (the p_μ are constants, so the hamiltonian is independent of x).
2. Under Lorentz transformations (the $J_{\mu\nu}$ are constants, so the hamiltonian must produce $\dot{S} = -\dot{L}$).

3. Under parity and time reversal (we will only consider representations (j, j) or $(j, 0) \oplus (0, j)$ and their combinations for the Lorentz part, so $\eta \sim \xi^\dagger \beta$ where β is the parity matrix).

The solution to the equations of motion have to be physically consistent. This can be translated into three physical requirements:

1. Physical trajectories are in Minkowski space-time at all times. We normalize the proper time to the length of the trajectory, $\dot{x}^2 = 1$.
2. Physical variables remain in the representation space chosen (the invariants labeling the representation have to be constants).
3. Finally, there is only one object in phase space (combined with item 3. above this will lead to charge conjugation and CPT [9]).

We will not analyze these issues in general, as they are well known [10] from relativistic particle theory.

As said above, to disentangle momentum from velocity, one needs a lagrangian linear in velocities. This singular case is best treated by the method of Fadeev and Jackiw [11] where the momenta are considered as independent variables in configuration space, obtaining the same results than the standard canonical treatment with lesser effort. We will use a generalization of the Proca lagrangian [2] for arbitrary spin

$$L = i\frac{\lambda}{2}(\bar{\xi}\dot{\xi} - \dot{\bar{\xi}}\xi) + \frac{1}{2}(p_\mu\dot{x}^\mu - \dot{p}_\mu x^\mu) - \bar{\xi}\beta_\mu p^\mu \xi \quad (12)$$

where β is the irreducible part of

$$\beta_\mu = \frac{1}{2s}(\gamma_\mu \otimes I \otimes \dots \otimes I + \dots + I \otimes \dots \otimes I \otimes \gamma_\mu) \quad (13)$$

and s is the spin label. The Euler Lagrange equations are:

$$\dot{x}^\mu = \bar{\xi}\beta^\mu \xi, \quad \dot{p}_\mu = 0, \quad i\lambda\dot{\xi} = \beta \cdot p \xi, \quad i\lambda\dot{\bar{\xi}} = -\bar{\xi}\beta \cdot p \quad (14)$$

In (12,14) λ is an arbitrary scalar constant with the dimensions of action. We have also taken the spinor momentum $\eta = i\lambda\bar{\xi}$ without loss of generality. Observe how momentum and velocity are decoupled, the latter being fixed by the spinor degrees of freedom alone.

The spin tensor, Pauli-Lubansky vector, and radius are

$$S^{\mu\nu} = -\lambda s \bar{\xi} \beta^{\mu\nu} \xi, \quad r^\mu = \frac{\lambda s}{p^2} \bar{\xi} \beta^{\mu\nu} \xi p_\nu, \quad W^\mu = \lambda s \bar{\xi} \beta^{*\mu\nu} \xi p_\nu \quad (15)$$

where $\beta^{\mu\nu} = is[\beta^\mu, \beta^\nu]$. After some algebra one arrives to the Poisson brackets

$$\begin{aligned} \{W_\mu, W_\nu\} &= \epsilon_{\mu\nu\rho\lambda} p^\rho W^\lambda, \quad \{W_\mu, r_\nu\} = \epsilon_{\mu\nu\rho\lambda} p^\rho r^\lambda, \\ \{r_\mu, r_\nu\} &= -\frac{1}{p^4} \epsilon_{\mu\nu\rho\lambda} p^\rho W^\lambda \end{aligned} \quad (16)$$

which show the close relation between these variables and the rotations and boosts of the Lorentz group. We can use the constant W^μ as giving the spin in the rest frame. Also, it is possible to give the spin tensor (8) in terms of p and r solely $S^{\mu\nu} = p^\mu r^\nu - p^\nu r^\mu - ip^2\{r^\mu, r^\nu\}$. Observe that r can be made to vanish and still the Poisson Bracket $\{r, r\}$ of (16) be finite. Therefore, spin is not a consequence of zitterbewegung. We may have no zitterbewegung ($r = 0$), but finite spin ($W \neq 0$). There is also a loose relation between the product of bilinears and the symplectic structure; here we are freed of the problems [12] that other approaches have with the simultaneous treatment [13] of classical and -the would be- quantum variables.

The solution of the equations of motion (14) will of course lead to a trajectory of the form (6). We get it explicitly with

$$X^\mu = x^\mu(0) - r^\mu(0), \quad f(\tau) = \frac{1}{p^2} \bar{\xi} \beta \cdot p \xi \tau, \quad (17)$$

$$r^\mu(\tau) = r^\mu(0) \cos\left(\frac{\sqrt{p^2}\tau}{\lambda s}\right) + \frac{\lambda s}{\sqrt{p^2}} \dot{r}^\mu(0) \sin\left(\frac{\sqrt{p^2}\tau}{\lambda s}\right) \quad (18)$$

where $\dot{r}^\mu(0) = d^{\mu\nu}(p) \bar{\xi} \beta_\nu \xi$ and $r^\mu(0)$ is given by (15). All the spinors in (18) are given by their initial values at $\tau = 0$ and $d^{\mu\nu}(p)$ is the projector orthogonal to p . Equations (2,18) reveal the presence of zitterbewegung through a radius joining the instantaneous centre of the particle ($X + fp$) to its true position x . It is the centre which follows the external trajectory along p of a spinless particle. The radius rapidly rotates around the centre (with the frequency $2\sqrt{p^2}c^2/\hbar = 1.5 \cdot 10^{21} s^{-1}$ for an electron and $\lambda = 1$) tracing an ellipse whose parameters depend solely of the spinorial degrees of freedom of the particle. It is also worth to note here that the particle does not spend any energy nor momentum in keeping zitterbewegung on. One would also expect that classical external perturbations would act adiabatically on this internal rotation. The momentum would change according to the applied external force as dictated by the Newton second law, but the particle would continue to wind at each instant around the momentum to form a helix embracing the external trajectory.

We will now show that the above picture is not consistent for charged particles with spin. The reason is simply that zitterbewegung produces a radiation field that should carry fourmomentum away from the free particle!. We first introduce the electromagnetic interaction by the minimal substitution rule $p \rightarrow p - eA$ in (12). Then we obtain the electromagnetic current $j^\mu = e \bar{\xi} \beta^\mu \xi = eu^\mu$. We now recall that we can use all the results of the electrodynamics of spinless particles that do not use of relations of the type $p^\mu = mu^\mu$. Our spinning free particle will radiate at a rate

$$\frac{dp_{rad}^\mu}{d\tau} = -\frac{2}{3} e^2 (\ddot{r}_\mu \ddot{r}^\mu) u^\mu \quad (19)$$

Some comments are in order here: First, radiation will not drain four momentum off the particle ($\dot{p} = 0$). Second, radiation will occur even in the “rest system” ($\vec{p} = 0$) of the charge. Third, radiation will not slow zitterbewegung, that proceeds at fixed frequency. The cure to these accumulation of catastrophes is to forbid the free particle to radiate at all. Being \vec{r} spacelike, this can be achieved only if $\ddot{r} = 0$. Inspecting (18) we see that in this case $r = 0$ also. This is the condition of parallelism that can be attained when the spinor is one of the eigenvectors of the operator $\beta \cdot p$. For instance, for $s = 1/2$, $r = 0$ implies $\gamma \cdot p \xi_{\pm} = \pm \sqrt{p^2} \xi_{\pm}$. It is consistent [14] to normalize $\bar{\xi}_{\pm} \xi_{\pm} = \pm 1$ and that when $p^0 < 0$, ξ_{-} represents an antiparticle of momentum $-p$. Therefore, the condition of no spontaneous emission leads to free particles being either particles or antiparticles, never a mixture of these. Second quantization is a natural outcome of the above requisites. In addition, one recovers the old form of the Galileo principle.

The switch on of the electromagnetic interaction begins to complicate this very simple behaviour of the free spinning charged particle. Zitterbewegung will start with its proper invariable frequency, but with a small growing amplitude as the external field feeds into a finite r (for instance, for $s = 1/2$ this will proceed by creating a finite ξ_{-} (ξ_{+}) component out of the initial ξ_{+} (ξ_{-})). Conversely, when exiting from the interaction region the particle, generally off the mass shell, will radiate till reaching the pure state. One may think that the description in terms of spinors is a too elaborated one to deal with the spinning particle, at least in the free case. There may be a strong temptation to disentangle the radius from the Minkowski space coordinates. This could be achieved by performing a canonical transformation from x to $z = x - r$ getting rid of the cumbersome vanishing radius. However it is easy to check that this new coordinate is not canonical; and the other way around: any canonical and covariant coordinate without the radius will acquire an imaginary part, which in addition turns out to be spin(-or) dependent. The lesson is clear: even if the kinematics of the spinning particle traces that of the spinless case, there are additional degrees of freedom present, and these have to be taken into account to implement the canonical formalism. In this sense, the study of spinning particles in ordinary (x, p) phase space is hopeless.

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